Coupled Pendulums Without The Small Angle Approximation

Fluxion Example Description

1 Background

First of all, we define some important quantities for the system:

- l = Distance from the pivot point of the pendulum to the point of attachment for the spring
- L = Length of the pendulum rod
- k = Spring constant
- m = Mass of the pendulum
- ϕ_1, ϕ_2 = Angle of deflection

The rod and the spring are considered to be massless. In addition, if both pendulums are in the rest position, no force acts through the spring (the spring is in its equilibrium length).
In order to get the equations of motion for the system we choose a torque approach. This is based on an analogy of Newton's second law.

$$\vec{M} = J \cdot \dot{\vec{\omega}} \tag{1}$$

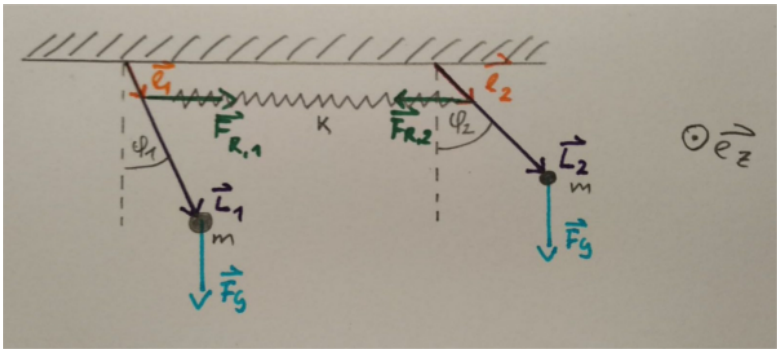
For reference:

$$\vec{F} = \dot{\vec{p}} = m \cdot \dot{\vec{v}} \tag{2}$$

Strictly speaking, a time-dependent mass m or a time-dependent moment of inertia would also cause a change in J, but in most cases these quantities are constant.
Vectorially, a torque M results from a force F acting at a point r from the pivot, which becomes:

$$\vec{M} = \vec{r} \times \vec{F} = r \cdot F \cdot sin(\phi) \cdot \hat{e} \tag{3}$$

Where e is a unit vector which is perpendicular to the vectors r and F.
The following sketch shows the position of the vectors used. The z-axis points out of the screen. The vector $\vec{e_z}$ is the unit vector in the z direction.



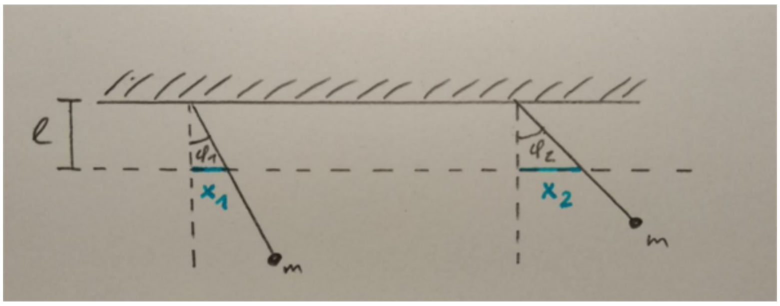
We now consider the acting torques on the left pendulum. The following applies:

$$\vec{M_1} = \vec{L_1} \times \vec{F_g} + \vec{l_1} \times \vec{F_{R,1}} = -m \cdot g \cdot L \cdot sin(\phi_1) \cdot \vec{e_z} + l \cdot F_{R,1} \cdot sin(\frac{\pi}{2} - \phi_1)\vec{e_z} \tag{4}$$

Where F_g is the weight of the point mass at the end of the rod and $F_{R,1}$ is the restoring force exerted by the spring. According to equation 3 this can be rewritten as already shown above. Please note that the angle between $F_{R,1}$ and l_1 is not ϕ_1, but instead $\frac{\pi}{2} - \phi_1$!
Now we have to determine $F_{R,1}$. Generally, this force is given by:

$$F_{R,1} = k \cdot x_{ges} \tag{5}$$

Where x_{ges} is the total displacement from the equilibrium position. It follows that this is the difference between the right pendulum's displacement x_2 and that of left pendulum x_1. The following sketch helps with determining these quantities: Assuming that the restoring force of



the spring always acts at a 90° angle (otherwise it will quickly become very complicated), we obtain for x_1 und x_2:

$$x_1 = l \cdot tan(\phi_1) \tag{6}$$

$$x_2 = l \cdot tan(\phi_2) \tag{7}$$

If we combine everything together we obtain:

$$J \cdot \dot{\vec{\omega_1}} = -m \cdot g \cdot L \cdot sin(\phi_1) \cdot \vec{e_z} + l^2 \cdot k \cdot (tan(\phi_2) - tan(\phi_1)) \cdot sin(\frac{\pi}{2} - \phi_1) \cdot \vec{e_z} \tag{8}$$

If we multiply this equation by $\vec{e_z}$ ($\dot{\vec{\omega_1}}$ points by definition in the $\vec{e_z}$ direction) and also divide by J, we obtain our first equation for the angular acceleration:

$$\dot{\omega_1} = (-m \cdot g \cdot L \cdot sin(\phi_1) + l^2 \cdot k \cdot (tan(\phi_2) - tan(\phi_1)) \cdot sin(\frac{\pi}{2} - \phi_1))/J \tag{9}$$

Analogously, the relationship for the second angular acceleration follows immediately. We just have to keep in mind that the restoring force now points in the opposite direction and accordingly exerts a torque in the other direction:

$$\dot{\omega_2} = (-m \cdot g \cdot L \cdot sin(\phi_2) - l^2 \cdot k \cdot (tan(\phi_2) - tan(\phi_1)) \cdot sin(\frac{\pi}{2} - \phi_2))/J \tag{10}$$

Now we can use the analog to the Newton machine for rotational motion, i.e. angular velocity is the rate of change of the angle and the rate of change of the angular velocity is the acceleration. Finally, we need the moment of inertia of a point mass (we assume that the mass is concentrated at the end of the rod):

$$J = m \cdot L^2 \tag{11}$$

2 What special cases can take place?

There are three different special cases to note for this system. A different oscillatory behavior then occurs depending on the starting conditions specified:

- **Synchronized Oscillations**: Both pendulums are given the same initial displacements ($\phi_1 = \phi_2$) - as expected, the spring is not stretched nor compressed and it becomes a (boring) normal pendulum oscillation.
- **Opposite Oscillations**: The two pendulums are provided with equal but opposite displacement angles ($\phi_1 = -\phi_2$), then the pendulums will oscillate with amplitudes of equal magnitude but in opposite directions.
- **The Beating Case**: This is the most interesting case. Only one of the pendulums is given an initial displacement ($\phi_1 = a, \phi_2 = 0$), so the pendulums transfer their energy back and forth during the course of the oscillation. (This is the default setting of the example)